Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ROOTS AND SINGULAR POINTS OF SEMI-ANALYTIC FUNCTIONS.

By Dunham Jackson.

Various theorems about the behavior of a function of several complex variables in the neighborhood of a point have analogous theorems corresponding to them for the case of a function which is analytic with respect to a particular one of its arguments, but merely continuous with regard to the whole set of variables, a function of the class which Bôcher has called semi-analytic.* It is the purpose of this note to point out some of the cases in which the analogy holds, and a simple case in which it is not preserved unimpaired. The method of treatment is closely related to that of an earlier paper by the author, on analytic functions.†

1. Weierstrass's factor theorem.

In proving Weierstrass's preparation theorem by means of Cauchy's integral formula,‡ the assumed analytic character of the function in question with regard to the totality of its arguments is used in such a way as to suggest that if a part of this analytic character were sacrificed in the hypothesis, nothing worse would result than a precisely corresponding weakening of the conclusion. By way of illustration, let us consider a function $f(x, y)$, defined throughout a region

$$(1) \qquad |x| \leqq h, \qquad |y| \leqq h,$$

surrounding the origin, the variable x being complex, and the variable y, for simplicity, being taken as real. Let it be assumed that if y is given any fixed value in the interval under consideration, the function f, with respect to its dependence on x alone, is analytic throughout the circle $|x| \leqq h$, while $f(x, y)$ is continuous in both arguments throughout the region (1). The function $f(x, 0)$ is assumed not to vanish identically, but to have a root, of order m let us say, at the point $x = 0$. It will be seen that $f(x, y)$ admits of factorization in the neighborhood of the origin, in accordance with a formula exactly like that which is ordinarily obtained for analytic functions.

---

* Bôcher, On semi-analytic functions of two variables, *Annals of Mathematics*, vol. 12 (1910–11), pp. 18–26.

† D. Jackson, Non-essential singularities of functions of several complex variables, *Annals of Mathematics*, vol. 17 (1915–16), pp. 172–179. This paper will be referred to simply by the letter A.

‡ Cf., e. g., Picard, Traité d'analyse, vol. 2, p. 244 of the first edition, 264 of the second edition.

Let $r_1 < h$ be a positive number such that $f(x, 0)$ does not vanish for $0 < |x| \leqq r_1$. The circle $|x| = r_1$, $y = 0$, is interior to a region of the space of the variables x, y, throughout which f is continuous in both variables and different from zero. The property of continuity extends to the derivative $f_x(x, y)$, as was pointed out by Bôcher.* Form the contour integral

$$\int_C \frac{f_x(x, y)}{f(x, y)}\, dx,$$

with the circle $|x| = r_1$ as path of integration. Since the integrand is continuous in both arguments, the value of the integral is a continuous function of y in the neighborhood of the value $y = 0$. On the other hand, it is recognized that for each particular value of y the integral gives the number of roots of the function $f(x, y)$ inside the circle $|x| \leqq r_1$, multiplied by $2\pi i$. The term "number of roots" is used for brevity to mean "sum of the orders of the roots." Since this number is always a positive integer and depends continuously on y, it must be constantly equal to m.

That is, it is possible to write down, for each particular value of y within a sufficiently restricted interval about the origin, a polynomial of the mth degree,

$$\psi(x, y) = x^m - p_1(y)x^{m-1} + p_2(y)x^{m-2} + \cdots \pm p_m(y), \tag{2}$$

whose roots coincide in location and multiplicity with those of $f(x, y)$. Nothing is said for the moment with regard to the character of the functions $p_r(y)$; the statement at present is merely the obvious one that such functions exist. The quotient

$$\varphi(x, y) = \frac{f(x, y)}{\psi(x, y)},$$

for each particular value of y, has only removable singularities as a function of x, and, if suitably defined at the points where these singularities occur, is analytic in x and different from zero throughout the circle $|x| \leqq r_1$. For x in this circle, then, and for y in a suitable interval $|y| \leqq r_2$, f has the representation

$$f(x, y) = \psi(x, y)\varphi(x, y), \tag{3}$$

where ψ has the form (2), and φ is everywhere different from zero. So much is immediate; it remains to discuss the continuity of the functions p_r and φ.

---

* Loc. cit., p. 20; the fact that the derivative is continuous in both variables may be recognized by expressing it as a contour integral, and making use of the continuity of the integrand.

Let C denote the same circle as before. The expression

$$\frac{1}{2\pi i}\int_C \frac{x^k f_x(x, y)}{f(x, y)}\, dx,$$

where k is any positive integer, gives the sum of the kth powers of the values of x in the circle for which $f(x, y)$ vanishes, each repeated according to the multiplicity of the corresponding root—for brevity, the sum of the kth powers of the roots. Each integral is a continuous function of y, because of the continuity of the integrand in both arguments. Since the sums of powers are continuous functions of y, the same thing is true of the elementary symmetric functions of the roots, $p_1(y), \cdots, p_m(y)$, which form the coefficients in (2).

Consider now the function φ, supposed to have been defined at its points of removable discontinuity so as to be analytic in x without exception. Its value can be written in the form

$$\varphi(x, y) = \frac{1}{2\pi i}\int_C \frac{\varphi(t, y)}{t - x}\, dt.$$

The integrand is continuous in t, x, and y, on the path of integration, if x is inside the circle and y is sufficiently small. Hence φ, obviously continuous in both its arguments except at the points where singularities were removed, is continuous there also.

The conclusion may be stated in the following form:

THEOREM I. *Let $f(x, y)$ be a function of the complex variable x and the real variable y, defined throughout a region containing the origin, analytic in x, and continuous in both variables together. Let $f(x, 0)$ vanish for $x = 0$, without vanishing identically. Then, throughout a sufficiently restricted neighborhood of the origin, $f(x, y)$ can be factored in accordance with equation* (3), *where $\varphi(x, y)$ is analytic in x, continuous in both variables, and everywhere different from zero, and $\psi(x, y)$ has the form* (2), *the coefficients $p_r(y)$ being continuous functions of y which vanish for $y = 0$.*

The validity of the very last clause is an immediate consequence of the definition of ψ.

A considerable part of the substance of this theorem, showing as it does that the roots of f depend continuously on y, must be regarded as well known; but the author does not remember having seen exactly the present formulation of it. A similar remark may apply to some of the later theorems.

Interpreted as a theorem on implicit functions, the assertion is, for $m = 1$, that if $f(x, y)$ is analytic in x and continuous in x and y in the neighborhood of the origin, if, further, $f(0, 0) = 0$, but $f_x(0, 0) \neq 0$, then the equation

$$f(x, y) = 0$$

defines x as a single-valued continuous function of y for sufficiently small values of y. It will be observed that there is no explicit hypothesis as to continuity of $f_x(x, y)$. In this connection, however, it must be taken into account that f is assumed to be continuous in both variables throughout a region in which x varies over a two-dimensional part of the complex plane; examples can readily be constructed to show that the theorem is not applicable when the continuity is granted merely for real values* of x and the assumed range of values of y. The distinction is analogous to that which is brought out by a Fourier's series such as

$$\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \cdots,$$

which is a uniformly convergent series of analytic functions, in a certain sense, but does not represent an analytic function.† It should be remarked, furthermore, that the continuity of $f_x(x, y)$ is implicitly provided for in the hypothesis as stated for a complex region, by virtue of the theorem of Bôcher already referred to.

It is obvious that Theorem I is capable of extension, with only the most superficial formal changes in statement and proof, to the case of a function $f(x, y_1, \cdots, y_n)$, with any number of real parameters. The assumed reality of the parameters is no restriction then, since a complex variable may be regarded as a pair of real variables.

A somewhat less trivial extension is obtained by including in the hypothesis an assumption that $f(x, y)$ can be differentiated q times with regard to y, and that the qth derivative is a continuous function of x and y together. Under these circumstances, the integrals concerned, whether in the proof of Bôcher's theorem about the derivative of f with regard to x, or in the reasoning of the present paper, can be differentiated q times with regard to y under the sign of integration, and the result of the differentiation will be a continuous function in each case. To give all the details of the reasoning would be a decidedly long process, but the individual steps are perfectly simple. The conclusion is as follows:

---

* E. g., let

$$f(x, y) = x \text{ for } y \leqq 0, \qquad f(x, y) = x + \frac{x}{y \log y} e^{-\frac{x^2}{y^2}} \text{ for } y > 0.$$

For the continuity, notice that

$$\left| \frac{x}{y \log y} e^{-\frac{x^2}{y^2}} \right| \leqq \frac{\gamma}{\log y},$$

if γ is the maximum of te^{-t^2} (all variables regarded as real). The equation $f(x, y) = 0$ gives

$$x = 0 \text{ for } y \leqq 0, \qquad x = 0 \text{ or } \pm y\sqrt{\log (1/y) - \log\log (1/y)} \text{ for } y > 0.$$

† Cf., e. g., Byerly, Fourier's series and spherical harmonics, pp. 39, 40.

THEOREM I*a*. *If $f(x, y)$ in Theorem I has a qth derivative with regard to y, which is continuous in both variables, then the functions $p_r(y)$ and $\varphi(x, y)$ also will have continuous qth derivatives with regard to y, the qth derivative of φ being continuous in both variables.*

This result also can be extended so as to take account of any derivative, of any order, with regard to any number of parameters* $y_1, \cdots, y_n$.

2. Singularity of the first kind.

If a function $f(x, y)$ is analytic in x, except for poles, for each particular value of y, the distribution of the poles is said to be continuous at a point (a, b), if the following condition is satisfied:† There exists a positive number r_1, as small as may be desired, with a corresponding positive number r_2, such that for $|y - b| \leqq r_2$ there are no poles on the boundary of the circle $|x - a| \leqq r_1$, and the sum of the orders of the poles inside the circle is constant. With regard to such functions, this assertion may be made:

THEOREM II. *Let $f(x, y)$ be a function of the complex variable x and the real variable y, having the following properties in the neighborhood of the origin:*

(*a*) *If y is held fast, $f(x, y)$ is an analytic function of x, except for poles.*

(*b*) *The distribution of these poles is continuous at the origin.*

(*c*) *Except at the points corresponding to poles, f is continuous in both variables together.*

Then f can be represented throughout a sufficiently restricted neighborhood of the origin, except at the singular points specified above, in the form

$$f(x, y) = \frac{\varphi(x, y)}{\psi(x, y)},$$

where φ and ψ are analytic in x and continuous in both variables throughout the neighborhood in question, and φ does not vanish there.

The denominator may be taken as a polynomial in x, with coefficients which are continuous functions of y.

The proof corresponds precisely to that of Theorem I of the paper A, and moreover closely resembles the proof of Theorem I of the present paper, so that it is unnecessary to give it here. The theorem can be extended at once to the case of several parameters, or to the case of functions having continuous derivatives of any order with regard to the parameter or parameters.

---

* It is readily seen that if $f(x, y)$ and its qth derivative with regard to y are continuous in both variables, the same is true of the intermediate derivatives with regard to y. A corresponding theorem does not hold for cross derivatives with regard to several parameters; but if the continuity of the intermediate derivatives is postulated explicitly, the reasoning can be carried through without difficulty.

† See the paper A, pp. 172–73.

3. Singularity of the second kind.

When the attempt is made to adapt Theorem II of the paper A to functions of the sort that we are considering here, essential differences between analytic and semi-analytic functions begin to manifest themselves. A theorem will be obtained, to be sure, with comparatively slight changes in conclusion and proof; but the necessity of the changes that are made must not be overlooked. The hypothesis will be as follows; the conclusion will be stated after an exposition of the reasoning by which it is established:

HYPOTHESIS OF THEOREM III. *Let $f(x, y)$ be a function of the complex variable x and the real variable y, having the following properties in the neighborhood of the origin:*

(*a*) *If y is held fast, $f(x, y)$ is an analytic function of x, except at most for poles.*

(*b*) *The distribution of these poles is continuous at every point except the origin itself.*

(*c*) *Except at the origin and at the points corresponding to poles, f is continuous in x and y together.*

The first step of the proof, in the analytic case, consists in showing that there is a region $|x| \leqq \rho_1$, $|y| \leqq \rho_2$, such that if y_0 is any particular point subject to the condition $0 < |y_0| \leqq \rho_2$, the number of poles of $f(x, y)$ in the circle $|x| \leqq \rho_1$ remains constant for all values of y in a sufficiently restricted neighborhood of the point y_0. So much is readily proved, in the same way, in the present case also. Then the inference is drawn, by means of the Heine-Borel theorem, that the number of poles in the circle is constant throughout the whole region $0 < |y| \leqq \rho_2$. It is at this point that a difference appears. For when the parameter y is restricted to real values, the condition $0 < |y| \leqq \rho_2$ does not define a single connected region, but a pair of distinct intervals, and there is no way of passing from one to the other. Taking note of the fact that the number of poles in the circle of the x-plane is constant for $0 < y \leqq \rho_2$ and constant also for $0 > y \geqq -\rho_2$, we will show that these two constant numbers may perfectly well be different.

Consider, for example, a function $f(x, y)$, defined as follows:

$$f(x, y) = \frac{x+y}{x-y} \text{ for } y > 0; \qquad f(x, y) = 1 \text{ for } y \leqq 0.$$

It satisfies all the conditions of the hypothesis, but has not the same number of poles for positive and for negative values of y. If this function is expressed in the form $f = \varphi/\psi$, where φ and ψ are analytic in x and continuous in both variables, then, by Theorem I, the number of values of x for which $\varphi = f\psi$ vanishes must remain constant when y varies slightly in either direction from the value $y = 0$. There must surely be a root for $y > 0$, so there must be one also for $y < 0$. When $y < 0$, however, $f = 1$, $\psi = \varphi$, and ψ also must have a root; and at these points where φ and ψ vanish simultaneously, the representation can not be said strictly to hold, although f itself is continuous.* As a matter of fact, it is obvious that f can be represented as a quotient φ/ψ, by setting

$$\varphi = x + y \text{ for } y \geqq 0, \qquad \varphi = x \text{ for } y \leqq 0,$$

$$\psi = x - y \text{ for } y \geqq 0, \qquad \psi = x \text{ for } y \leqq 0.$$

It will appear that this simple example is typical of what can, as well as of what can not, be expected to hold in general.

Let us return to the consideration of a general function $f(x, y)$ satisfying the hypothesis. The condition on the number ρ_1, mentioned above, is merely† that $f(x, 0)$ be analytic for $|x| = \rho_1$. It may be supposed further that ρ_1 is chosen so small that $f(x, 0)$ has no pole nor root‡ for $0 < x \leqq \rho_1$. We have seen that the number of poles of $f(x, y)$ in the circle $0 \leqq x \leqq \rho_1$ is constant, let us say equal to m, for $0 < y \leqq \rho_2$. For the present, let y be restricted to this interval, closed, on occasion, by the value $y = 0$; similar reasoning will of course apply to the interval $0 \geqq y \geqq -\rho_2$, considered by itself. The circle in the x-plane will be denoted by R_1.

In the continuation of the proof for the analytic case, it is shown that $p_1(y), \cdots, p_m(y)$, the elementary symmetric functions of the poles in R_1, are analytic functions of y for $0 < |y| \leqq \rho_2$. The proof is given by means of contour integrals extended around small circles interior to R_1. It is clear from the definition of the functions $p_r(y)$ that they remain finite as y approaches 0, and so it is inferred that if suitably defined for $y = 0$ they are analytic there. The part of the reasoning which deals with the contour integrals can be adapted to the present hypothesis, and shows that each $p_r(y)$ is continuous for $0 < y \leqq \rho_2$. But even though $p_r(y)$

---

* A similar remark would apply to a representation in which φ and ψ vanish identically for $y = 0$, although Theorem I could not be used.

† It is naturally assumed throughout that the variables are restricted to the region for which the hypothesis holds.

‡ This is impossible in the special case, not excluded by the hypothesis, that $f(x, 0)$ vanishes identically. But the single passage in the subsequent reasoning where use is made of the assumption that there is no root for $0 < x \leqq \rho_1$, becomes entirely superfluous if $f(x, 0)$ does not have a pole at the origin.

does remain finite as y approaches zero, there is of course no general theorem for functions of a real variable which makes it possible to extend the continuity to the point $y = 0$. It is to be shown by independent reasoning that $\lim_{y=0} p_r(y) = 0$. The proof is almost immediate, as soon as its necessity has been pointed out.

Let ϵ be an arbitrarily small positive quantity. By the condition imposed on ρ_1, $f(x, 0)$ is analytic for $\epsilon \leqq x \leqq \rho_1$. It follows from the hypothesis of continuity in the distribution of the poles, that $f(x, y)$ is analytic in x throughout the same region, if y is sufficiently small. That is, for small positive values of y, all m of the poles which $f(x, y)$ must have in R_1 are within the circle $|x| \leqq \epsilon$. This means that

$$|p_1(y)| < m\epsilon, \qquad |p_2(y)| < \frac{m(m-1)\epsilon^2}{2},$$

and so on. If the functions $p_r(y)$ are set equal to 0 for $y = 0$, they are continuous for $0 \leqq y \leqq \rho_2$.

Let

$$\psi(x, y) = x^m - p_1(y)x^{m-1} + \cdots \pm p_m(y), \qquad \varphi(x, y) = f(x, y)\psi(x, y).$$

In the case of analytic functions, Cauchy's integral formula is used to prove that after the elimination of removable singularities, φ is analytic in both variables, except at the single point (0, 0); and since an isolated singularity of an analytic function of two variables must be removable, φ can be made analytic without exception. Once more the extension to semi-analytic functions requires a little care. In the application of Cauchy's integral formula there is no difficulty; it is recognized that φ is continuous in x and y together, as well as analytic in x, if removable singularities are disposed of and the point (0, 0) is left out of account. But for this last point special considerations are required.

The function $f(x, 0)$ may be analytic at the origin, or it may have a pole there. In the latter case, the order of the pole will not in general be equal to m (can not be equal to m, if the distribution of the poles is actually discontinuous at the origin). The essential thing is, that a pole of order higher than m is impossible. Consider the integral of f_x/f, extended around the circle $|x| = \rho_1$. It is a continuous function of y throughout the neighborhood of the value $y = 0$. It gives the difference between the number of poles and the number of roots inside the circle, each counted according to its multiplicity. Therefore, being continuous, it is constant. For $y > 0$, the number of poles is m, the number of roots is essentially positive or zero, and the difference can not exceed m. For $y = 0$, there are no roots at all, and the difference, still having the same value as before, is now simply the number of poles, in other words, the order of the single pole that remains. Thus the last difficulty is disposed of. The function $\psi(x, 0)$ reduces to the single term x^m. The product φ has at most a removable singularity at the origin. When this is closed up, φ is analytic in x, and, by an application of Cauchy's integral formula once more, continuous in x and y together.

The essential conclusion is, that under the conditions of the hypothesis, f can be expressed in the form φ/ψ, where φ and ψ are semi-analytic for $0 \leqq |x| \leqq \rho_1$, $0 \leqq y \leqq \rho_2$, the denominator vanishing only at singular points of f.

The only unexpected feature here, if it may be called unexpected, is the restriction of y to a one-sided neighborhood of the origin. For the interval $0 \geqq y \geqq -\rho_2$, as has already been indicated, a similar conclusion can be drawn. Let m_1 be the number of poles for $y < 0$, the degree of the corresponding polynomial ψ. If it happens that $m = m_1$, both polynomials ψ join on continuously to the function x^m for $y = 0$. In this case, ψ is semi-analytic, and φ can be made semi-analytic, throughout the whole neighborhood $0 \leqq |x| \leqq \rho_1$, $0 \leqq |y| \leqq \rho_2$. Otherwise, suppose $m > m_1$. If

$$\psi_1(x, y) = \psi(x, y) \text{ for } y > 0, \qquad \psi_1(x, y) = x^{m-m_1}\psi(x, y) \text{ for } y < 0,$$

$$\psi_1(x, 0) = x^m,$$

then ψ_1 is semi-analytic throughout the neighborhood of the origin, $\varphi_1 = f\psi_1$ is semi-analytic except for removable singularities, and f is equal to φ_1/ψ_1, in which representation the denominator may vanish at points which are not singular points of f. The analogy between analytic and semi-analytic functions is perhaps upheld as far as possible in the following enunciation:

THEOREM III. *Under the hypothesis already stated, f can be represented throughout a sufficiently restricted neighborhood of the origin, except at the singular points specified, and possibly one other point for each positive or each negative value of y, in the form*

$$f(x, y) = \frac{\varphi(x, y)}{\psi(x, y)},$$

where φ and ψ are analytic in x and continuous in both variables throughout the neighborhood in question.

It may be remarked, without details of proof, that the restrictive phrase "and possibly one other point . . ." can be dispensed with, even when $m \neq m_1$, provided only that neither m nor m_1 is zero; for the extraneous factor in one of the functions ψ can be so chosen as not to introduce any new root, but merely to increase the multiplicity of one already present.

When it is attempted to extend this theorem, as earlier theorems were extended, by supposing the functions to possess continuous derivatives with regard to y, the analogy fails still further. Consider, for example, the function*

$$f(x, y) = \frac{y}{x - \sqrt{y}}, \qquad f(0, 0) = 0.$$

When y is held fast, it is an analytic function of x, except for a single pole when $y \neq 0$. The situation of the pole varies continuously with y, except for $y = 0$, where it ceases to exist. The function is continuous in x and y together, and has a derivative with regard to y,

$$f_y = \frac{x - \frac{1}{2}\sqrt{y}}{(x - \sqrt{y})^2},$$

which is continuous in both variables, at all points except those where the denominator vanishes. Nevertheless, the function $p_1(y) = \sqrt{y}$ does not have a continuous derivative, even for the one-sided neighborhood $0 \leqq y \leqq \delta$. A representation as a quotient of two functions with continuous derivatives can be obtained by setting $\varphi = y^2$, $\psi = xy - y\sqrt{y}$.

It is obvious that the discussion could be carried further, but enough has been said to illustrate the possibilities of the theory and its limitations. In particular, it may be remarked that a semi-analytic function of one complex and two real variables, unlike an analytic function of three complex variables, can have a singularity of the sort considered in Theorem III, with no other singularities of the same sort in its neighborhood, e. g.,

$$f(x, y, z) = \frac{x^2 + y^2 + z^2}{x^2 - y^2 - z^2}, \qquad f(0, 0, 0) = 1 \qquad (y \text{ and } z \text{ real}).$$

The neighborhood of the origin in the plane of the parameters is now a connected region, and for such functions, Theorem III resumes a form more closely resembling the analogous theorem for analytic functions of two complex variables.

Harvard University,
Cambridge, Mass.

---

* To admit negative values of y, let $f(x, y) = \dfrac{y}{x - \sqrt{|y|}}$.